

REGULAR ORBITS OF SYMMETRIC SUBGROUPS ON PARTIAL FLAG VARIETIES

DAN CIUBOTARU, KYO NISHIYAMA, AND PETER E. TRAPA

1. INTRODUCTION

The main result of the current paper is a new parametrization of the orbits of a symmetric subgroup K on a partial flag variety \mathcal{P} . The parametrization is in terms of certain Spaltenstein varieties, on one hand, and certain nilpotent orbits, on the other. One of our motivations, as explained below, is related to enumerating special unipotent representations of real reductive groups. Another motivation is understanding (a portion of) the closure order on the set of nilpotent coadjoint orbits.

In more detail, suppose G is a complex connected reductive algebraic group and let θ denote an involutive automorphism of G . Write K for the fixed points of θ , and \mathcal{P} for a variety of parabolic subalgebras of a fixed type in \mathfrak{g} , the Lie algebra of G . Then K acts with finitely many orbits on \mathcal{P} , and these orbits may be parametrized in a number of ways (e.g. [M], [RS], [BH]), each of which may be viewed as a generalization of the classical Bruhat decomposition. (This latter decomposition arises if $G = G_1 \times G_1$, θ interchanges the two factors, and \mathcal{P} is taken to be the full flag variety of (pairs of) Borel subalgebras.) We give our parametrization of $K \backslash \mathcal{P}$ in Corollary 2.14 and then turn to applications and examples in later sections.

As mentioned above, one of the applications we have in mind concerns the connection with nilpotent coadjoint orbits for K . To each orbit $Q = K \cdot \mathfrak{p}$ of parabolic subalgebras in \mathcal{P} , we obtain such a coadjoint orbit as follows. Let \mathfrak{k} denote the Lie algebra of K , and consider

$$(1.1) \quad K \cdot [(\mathfrak{g}/\mathfrak{p})^* \cap (\mathfrak{g}/\mathfrak{k})^*] = K \cdot (\mathfrak{g}/(\mathfrak{p} + \mathfrak{k}))^* \subset \mathfrak{g}^*;$$

here and elsewhere we implicitly invoke the inclusion of $(\mathfrak{g}/\mathfrak{p})^*$ and $(\mathfrak{g}/\mathfrak{k})^*$ into \mathfrak{g}^* and take the intersection there. Suppose for simplicity K is connected. Then the space in (1.1) is irreducible. It also consists of nilpotent elements and is K invariant. Since the number of nilpotent K orbits on $(\mathfrak{g}/\mathfrak{k})^*$ is finite [KR], the space must contain a unique dense K orbit, call it $\Phi_{\mathcal{P}}(Q)$. (It is easy to adapt this argument to yield the same conclusion if K is disconnected.) Thus we obtain a natural map

$$(1.2) \quad \Phi = \Phi_{\mathcal{P}} : K \backslash \mathcal{P} \longrightarrow K \backslash \mathcal{N}_{\mathcal{P}}^{\theta},$$

where $\mathcal{N}_{\mathcal{P}}^{\theta}$ denotes the cone of nilpotent elements in

$$(1.3) \quad [G \cdot (\mathfrak{g}/\mathfrak{p})^*] \cap (\mathfrak{g}/\mathfrak{k})^*.$$

In fact, the map $\Phi_{\mathcal{P}}$ is the starting point of our parametrization of $K \backslash \mathcal{P}$ in Section 2. For orientation, in the setting of the Bruhat decomposition mentioned above, the map may be interpreted as taking Weyl group elements to nilpotent coadjoint orbits. (Concretely it amounts to taking an element w to the dense orbit in the G_1 saturation of the intersection of the nilradicals of two Borel subalgebras in relative position w .)

Just as the Bruhat order on a Weyl group is easier to understand than the classification and closure order on nilpotent orbits, the set of K orbits on \mathcal{P} in some sense behaves more nicely than the set of K orbits on $\mathcal{N}_{\mathcal{P}}^{\theta}$. The former (and the closure order on it) can be described uniformly, for instance [RS]. This is not the case for $K \backslash \mathcal{N}_{\mathcal{P}}^{\theta}$, where any (known) classification involves at least some case-by-case analysis. So a natural question becomes: can one translate the uniform features of K orbits on \mathcal{P} to the setting of K orbits on $\mathcal{N}_{\mathcal{P}}^{\theta}$ using $\Phi_{\mathcal{P}}$? This is the viewpoint we adopt in Section

2. In particular, one may ask the following: given a K orbit \mathcal{O}_K in $\mathcal{N}_{\mathcal{P}}^{\theta}$, does there exist a canonical element Q of $K \backslash \mathcal{P}$ such that $\Phi_{\mathcal{P}}(Q) = \mathcal{O}_K$? If so, we would be able to embed the set of K orbits on $\mathcal{N}_{\mathcal{P}}^{\theta}$ into (the more uniformly behaved) set of K orbits on \mathcal{P} . One might optimistically hope to understand a parametrization of $K \backslash \mathcal{N}_{\mathcal{P}}^{\theta}$ (and understand its closure order) in this way.

The simplest way to produce affirmative answer to this last question is if the fiber of $\Phi_{\mathcal{P}}$ over \mathcal{O}_K consists of a single element Q . So it is desirable to have a formula for the cardinality of the fiber. Using ideas of Rossmann and Borho-MacPherson, we give such a formula in Proposition 2.10 in terms of certain Springer representations. The question of whether the fiber consists of a single element then becomes a multiplicity one question about certain Weyl group representations. We then turn to two natural questions:

- (1) Can one find a natural class of orbits \mathcal{O}_K for which the fiber $\Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$ is indeed a singleton?
- (2) If so, can one give an *effective* algorithm to determine the fiber? (This is clearly important if one really wants to use these ideas to try to classify K orbits on $\mathcal{N}_{\mathcal{P}}^{\theta}$ uniformly.)

We give affirmative answers to these questions in Proposition 3.7 and Remark 3.10 respectively. The class of K orbits we find are those \mathcal{O}_K such that $\mathcal{O} = G \cdot \mathcal{O}_K$ is an *even* complex orbit; then $\Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$ consists of a single element if \mathcal{P} is taken to be the partial flag variety such that $T^*\mathcal{P}$ is a resolution of singularities of the closure of \mathcal{O} . (The corresponding K orbits on \mathcal{P} are the regular orbits of the title.) Perhaps surprisingly the algorithm answering (2) relies on the Kazhdan-Lusztig-Vogan algorithm [V1] for computing the intersection homology groups (with coefficients) of K orbit closures on the full flag variety.

The setting of Section 3 may appear too restrictive to be of much practical value. But in Section 4 we recall that it is exactly the geometric setting of the Adams-Barbasch-Vogan definition of Arthur packets. More precisely, since the ground field is \mathbb{C} , θ arises as the complexification of a Cartan involution for a real form $G_{\mathbb{R}}$ of G . We show that the algorithm of Remark 3.10 gives an effective means to compute a distinguished constituent of each Arthur packet of integral special unipotent representations for $G_{\mathbb{R}}$. According to the Arthur conjectures, these representations should be unitary. This is a striking prediction (which is still open in general), since the constructions leading to their definition have nothing to do with unitarity.

Section 4 is highly technical unfortunately, but we have included it in the hope that it is perhaps more accessible than [ABV, Chapter 27] (upon which it is of course based). We have also included it for another reason which is easy to understand from the current context. If it were possible to give affirmative answers to questions (1) and (2) above to a wider class of orbits than we consider in Section 3, then the ideas of Section 4 translate those answers into new conclusions about special unipotent representations of real reductive groups. In recent joint work with Barbasch, one of us (PT) has made progress in this direction. The precise formulation of these results involves a rather different set of ideas, and the details will appear elsewhere.

Finally, in Section 5, we consider a number of examples illustrating some subtleties of the parametrization of Section 2.

Acknowledgements. KN and PT would like to thank the Hausdorff Research Institute for Mathematics for its hospitality during their stay in 2007. They would also like to thank the organizers of the joint MPI-HIM program devoted to representation theory, complex analysis and integral geometry. KN is partially supported by JSPS Grant-in-Aid for Scientific Research (B) #17340037. PT was supported by NSA grant MSPF-06Y-096 and NSF grant DMS-0532393.

2. PARAMETRIZING $K \backslash \mathcal{P}$

The main result of this section is Corollary 2.14 which gives a parametrization of the K orbits on \mathcal{P} . As Propositions 2.10 and 2.15 show, the parametrization is closely related to Springer's Weyl group representations.

We begin with a discussion of the set $K \backslash \mathcal{B}$ of K orbits on \mathcal{B} , the full flag variety of Borel subalgebras in our fixed complex reductive Lie algebra \mathfrak{g} . Basic references for this material are [M] or [RS]. The set $K \backslash \mathcal{B}$ is partially ordered by the inclusion of orbit closures. It is generated by closure relations in codimension one. We will need to distinguish two kinds of such relations. To do so, we fix a base-point $\mathfrak{b}_\circ \in \mathcal{B}$ and a Cartan \mathfrak{h}_\circ in \mathfrak{b}_\circ . We write $\mathfrak{b}_\circ = \mathfrak{h}_\circ \oplus \mathfrak{n}_\circ$ for the corresponding Levi decomposition, and let $\Delta^+ = \Delta^+(\mathfrak{h}_\circ, \mathfrak{n}_\circ)$ denote the roots of \mathfrak{h}_\circ in \mathfrak{n}_\circ . For a simple root $\alpha \in \Delta^+$, let \mathcal{P}_α denote the set of parabolic subalgebras of type α , and write π_α for the projection $\mathcal{B} \rightarrow \mathcal{P}_\alpha$.

Fix K orbits Q and Q' on \mathcal{B} . If K is connected, then Q is irreducible, and hence so is $\pi_\alpha^{-1}(\pi_\alpha(Q))$. Thus $\pi_\alpha^{-1}(\pi_\alpha(Q))$ contains a unique dense K orbit. In general, K need not be connected and Q need not be irreducible. But it is easy to see that the similar reasoning applies to conclude $\pi_\alpha^{-1}(\pi_\alpha(Q))$ always contains a dense K orbit. We write $Q \xrightarrow{\alpha} Q'$ if

$$\dim(Q') = \dim(Q) + 1$$

and

$$Q' \text{ is dense in } \pi_\alpha^{-1}(\pi_\alpha(Q)).$$

This implies that Q is codimension one in the closure of Q' . The relations $Q < Q'$ for $Q \xrightarrow{\alpha} Q'$ do not generate the full closure order however. Instead we must also consider a kind of saturation condition. More precisely, whenever a codimension one subdiagram of the form

$$(2.1) \quad \begin{array}{ccc} & Q_1 & \\ \alpha \nearrow & & \nwarrow \alpha \\ Q_2 & & Q_3 \\ & Q_4 & \\ & \nwarrow \alpha & \nearrow \alpha \end{array}$$

is encountered, we complete it to

$$(2.2) \quad \begin{array}{ccc} & Q_1 & \\ \alpha \nearrow & & \dashrightarrow \alpha \\ Q_2 & & Q_3 \\ & Q_4 & \\ & \nwarrow \alpha & \nearrow \alpha \end{array}$$

New edges added in this way are dashed in the diagrams below. Note that this operation must be applied recursively, and thus some of the edges in the original diagram (2.1) may be dashed as the recursion unfolds. Following the terminology of [RS, 5.1], we call the partially ordered set determined by the solid edges the weak closure order.

Now fix a variety of parabolic subalgebras \mathcal{P} of an arbitrary fixed type and write $\pi_{\mathcal{P}}$ for the projection from \mathcal{B} to \mathcal{P} . For definiteness fix $\mathfrak{p}_\circ \in \mathcal{P}$ containing \mathfrak{b}_\circ , and write $\mathfrak{p}_\circ = \mathfrak{l}_\circ \oplus \mathfrak{u}_\circ$ for the Levi decomposition such that $\mathfrak{h}_\circ \subset \mathfrak{l}_\circ$. Then $K \backslash \mathcal{P}$ may be parametrized from a knowledge of the weak closure on $K \backslash \mathcal{B}$ as follows. Consider the relation $Q \sim_{\mathcal{P}} Q'$ if $\pi_{\mathcal{P}}(Q) = \pi_{\mathcal{P}}(Q')$; this is generated by the relations $Q \sim Q'$ if $Q \xrightarrow{\alpha} Q'$ for α simple in $\Delta(\mathfrak{h}_\circ, \mathfrak{l}_\circ)$. Equivalence classes in $K \backslash \mathcal{B}$ clearly are in bijection with $K \backslash \mathcal{P}$. (See also the parametrization of [BH, Section 1], especially Proposition 4.) Fix an equivalence class C and fix a representative $Q \in C$. The same reasoning that shows that $\pi_\alpha^{-1}(\pi_\alpha(Q))$ contains a unique dense K orbit also shows that

$$\pi_{\mathcal{P}}^{-1}(\pi_{\mathcal{P}}(Q))$$

contains a unique dense K orbit $Q_C \in K \backslash \mathcal{B}$. In other words, Q_C is the unique largest dimensional orbit among the elements in C . In fact Q_C is characterized among the elements of C by the condition

$$(2.3) \quad \dim \pi_\alpha^{-1}(\pi_\alpha(Q_C)) = \dim(Q_C)$$

for all α simple in $\Delta(\mathfrak{h}_o, \mathfrak{l}_o)$. It follows that the full closure order on $K \backslash \mathcal{P}$ is simply the restriction of the full closure order on $K \backslash \mathcal{B}$ to the subset of all maximal-dimensional representatives of the form Q_C . By restricting only the weak closure order, we may speak of the weak closure order on $K \backslash \mathcal{P}$.

We next place the map $\Phi_{\mathcal{P}}$ of (1.2) in a more natural context. Consider the cotangent bundle $T^*\mathcal{P} \subset \mathcal{P} \times \mathfrak{g}^*$. It consists of pairs (\mathfrak{p}, ξ) with

$$(2.4) \quad \xi \in T_{\mathfrak{p}}^*\mathcal{P} \simeq (\mathfrak{g}/\mathfrak{p})^*$$

The moment map $\mu_{\mathcal{P}}$ from $T^*\mathcal{P}$ to \mathfrak{g}^* maps a point (\mathfrak{p}, ξ) in $T^*\mathcal{P}$ simply to ξ . Consider now the conormal variety for K orbits on \mathcal{P} ,

$$T_K^*\mathcal{P} = \bigcup_{Q \in K \backslash \mathcal{P}} T_Q^*\mathcal{P},$$

where $T_Q^*\mathcal{P}$ denotes the conormal bundle to the K orbit Q . (In the special case $G = G_1 \times G_1$ and $\mathcal{P} = \mathcal{B}$ mentioned in the introduction, the conormal variety is the usual Steinberg variety of triples; see, for instance, the exposition of [DR].) In general we may identify

$$(2.5) \quad T_Q^*\mathcal{P} = \{(\mathfrak{p}, \xi) \mid \mathfrak{p} \in Q, \xi \in (\mathfrak{g}/(\mathfrak{k} + \mathfrak{p}))^*\},$$

and hence the image of $T_K^*\mathcal{P}$ under $\mu_{\mathcal{P}}$ is simply $\mathcal{N}_{\mathcal{P}}^{\theta}$. Moreover, the image of $T_Q^*\mathcal{P}$ under $\mu_{\mathcal{P}}$ is nothing but the space in (1.1). Hence $\Phi_{\mathcal{P}}(Q)$ is simply the unique dense K orbit in the moment map image of $T_Q^*\mathcal{P}$.

Here are some elementary properties of $\Phi_{\mathcal{P}}$.

Proposition 2.6. (1) *Fix $Q \in K \backslash \mathcal{P}$ and suppose $Q' \in K \backslash \mathcal{B}$ is dense in $\pi_{\mathcal{P}}^{-1}(Q)$. Then*

$$\Phi_{\mathcal{B}}(Q') = \Phi_{\mathcal{P}}(Q).$$

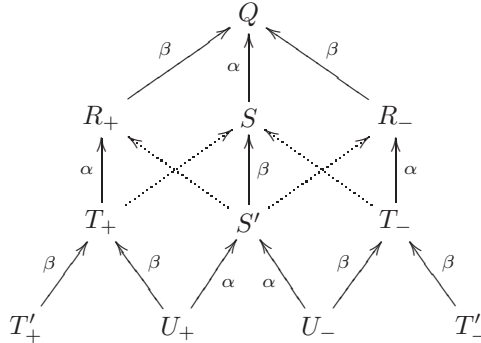
(2) *The map $\Phi_{\mathcal{P}}$ is order reversing from the weak closure order in $K \backslash \mathcal{P}$ to the closure order on $K \backslash \mathcal{N}_{\mathcal{P}}^{\theta}$; that is, if $Q < Q'$ in the weak closure order on $K \backslash \mathcal{P}$, then*

$$\overline{\Phi_{\mathcal{P}}(Q)} \supset \Phi_{\mathcal{P}}(Q').$$

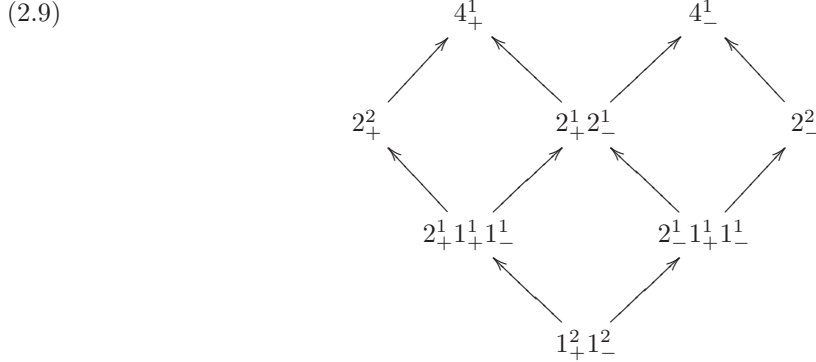
Proof. Part (1) is clear from the definitions. Part (2) reduces to the assertion for $Q \xrightarrow{\alpha} Q'$. In that case, it amounts to a rank one calculation where it is obvious. \square

Example 2.7. Proposition 2.6(2) fails for the full closure order on $K \backslash \mathcal{P}$. The first example which exhibits this failure is $G_{\mathbb{R}} = \mathrm{Sp}(4, \mathbb{R})$ and $\mathcal{P} = \mathcal{B}$. Let α denote the short simple root in Δ^+ and β the long one. The closure order for $K \backslash \mathcal{B}$ is as in the diagram in (2.8). Orbits on the same row of the diagram below all have the same dimension. (The bottom row consists of orbits of dimension one, the next row consists of orbits of dimension two, and so on.) Dashed lines represent relations in the full closure order which are not in the weak order.

(2.8)



Adopt the parametrization of $K \setminus \mathcal{N}^\theta$ given in [CM, Theorem 9.3.5] in terms of signed tableau. Let $(i_1)_{\epsilon_1}^{j_1} (i_2)_{\epsilon_2}^{j_2} \cdots$ denote the tableau with j_k rows of length i_k beginning with sign ϵ_k for each k . Then the closure order on $K \setminus \mathcal{N}^\theta$ is given by



Then Φ_B maps Q to $1_+^2 1_-^2$; R_\pm to $2_\pm^1 1_+^1 1_-^1$; S and S' to $2_+^1 2_-^1$; T_\pm and T'_\pm to 2_\pm^2 ; and U_\pm to 4_\pm^1 . Note that Φ_B reverses all closure relations *except* the two dashed edges indicating $T_\pm \subset \overline{S}$.

We are now in a position to determine the size of the fiber $\Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$ for $\mathcal{O}_K \in K \setminus \mathcal{N}_{\mathcal{P}}^\theta$. For $\xi \in \mathcal{O}_K$, let $A_K(\xi)$ (resp. $A_G(\xi)$) denote the component group of the centralizer in K (resp. G) of ξ . Obviously there is a natural map

$$A_K(\xi) \rightarrow A_G(\xi)$$

which we often invoke implicitly. Write $\mathrm{Sp}(\xi)$ for the Springer representation of $W \times A_G(\xi)$ on the top homology of the Springer fiber over ξ (normalized so that $\xi = 0$ gives the sign representations of W). Let

$$\mathrm{Sp}(\xi)^{A_K} = \mathrm{Hom}_{A_K(\xi)}(\mathrm{Sp}(\xi), \mathbb{1}).$$

Proposition 2.10. *Fix $\xi \in \mathcal{O}_K$. Then*

$$\begin{aligned} \#\Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K) &= \dim \mathrm{Hom}_{W(\mathcal{P})}(\mathrm{sgn}, \mathrm{Sp}(\xi)^{A_K}) \\ &= \dim \mathrm{Hom}_W(\mathrm{ind}_{W(\mathcal{P})}^W(\mathrm{sgn}), \mathrm{Sp}(\xi)^{A_K}). \end{aligned}$$

Proof. The second equality follows by Frobenius reciprocity. For the first, set

$$S_{\mathcal{P}} = \{Q \in K \setminus \mathcal{B} \mid Q \text{ is dense in } \pi_{\mathcal{P}}^{-1}(\pi_{\mathcal{P}}(Q))\}.$$

According to the discussion around (2.3) and Proposition 2.6(1), $\pi_{\mathcal{P}}$ implements a bijection

$$S_{\mathcal{P}} \cap \Phi_B^{-1}(\mathcal{O}_K) \rightarrow \Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K).$$

We will count the left-hand side if K is connected. If K is disconnected, there are a few subtleties (none of which are very serious) which are best treated later.

Consider the top integral Borel-Moore homology of the conormal variety $T_K^* \mathcal{P}$. Since we have assumed K is connected, the closures of the individual conormal bundles exhaust the irreducible components of $T_K^* \mathcal{P}$, and their classes form a basis of the homology,

$$H_{\mathrm{top}}^\infty(T_K^* \mathcal{P}, \mathbb{Z}) = \bigoplus_{Q \in K \setminus \mathcal{P}} [\overline{T_Q^* \mathcal{P}}].$$

If $\mathcal{P} = \mathcal{B}$, Rossmann [R] (extending earlier work of Kazhdan-Lusztig [KL]) described a construction giving an action of the Weyl group W on this homology space. The action is graded in the following sense that if $Q \in \Phi_B^{-1}(\mathcal{O}_K)$, then

$$w \cdot [\overline{T_Q^* \mathcal{B}}]$$

is a linear combination of conormal bundles to orbits in fibers $\Phi_B^{-1}(\mathcal{O}'_K)$ with $\mathcal{O}'_K \subset \overline{\mathcal{O}_K}$. Hence if we set

$$\Phi_B^{-1}(\mathcal{O}_K, \leq) = \bigcup_{\mathcal{O}'_K \subseteq \overline{\mathcal{O}_K}} \Phi_B^{-1}(\mathcal{O}'_K)$$

and

$$\Phi_B^{-1}(\mathcal{O}_K, <) = \bigcup_{\mathcal{O}'_K \subsetneq \overline{\mathcal{O}_K}} \Phi_B^{-1}(\mathcal{O}'_K)$$

then

$$\mathbf{M}(\mathcal{O}_K) := \bigoplus_{Q \in \Phi_B^{-1}(\mathcal{O}_K, \leq)} [\overline{T_Q^* \mathcal{B}}] \bigg/ \bigoplus_{Q \in \Phi_B^{-1}(\mathcal{O}_K, <)} [\overline{T_Q^* \mathcal{B}}]$$

is a W module with basis indexed by $\Phi_B^{-1}(\mathcal{O}_K)$. Rossmann's construction shows that

$$\mathbf{M}(\mathcal{O}_K) \simeq \mathrm{Sp}(\xi)^{A_K},$$

where $\xi \in \mathcal{O}_K$ as above. This proves the proposition for $\mathcal{P} = \mathcal{B}$. For the general case, we must identify $S_{\mathcal{P}}$ in terms of the Weyl group action. It follows from Rossmann's constructions that

$$s_{\alpha} \cdot [\overline{T_Q^* \mathcal{B}}] = -[T_Q^* \mathcal{B}]$$

if and only if

$$\dim \pi_{\alpha}^{-1}(\pi_{\alpha}(Q)) = \dim(Q).$$

Thus (2.3) implies that $S_{\mathcal{P}} \cap \Phi_B^{-1}(\mathcal{O}_K)$ indexes exactly the basis elements of $\mathbf{M}(\mathcal{O}_K)$ which transform by the sign representation of the Weyl group of type \mathcal{P} . The proposition thus follows in the case of K connected. (A complete proof in the disconnected case is discussed after Proposition 2.15.) \square

The above proof is extrinsic, in the sense that it is deduced from a statement about the $\mathcal{P} = \mathcal{B}$ case. We may argue more intrinsically (without reference to \mathcal{B}) using results of Borho-MacPherson [BM] as follows.

Fix $\xi \in \mathcal{N}_{\mathcal{P}}^{\theta}$ and consider $\mu_{\mathcal{P}}^{-1}(\xi)$. In terms of the identification around (2.4),

$$\mu_{\mathcal{P}}^{-1}(\xi) = \{(\mathfrak{p}, \xi) \mid \xi \in (\mathfrak{g}/\mathfrak{p})^*\}.$$

(Borho-MacPherson write \mathcal{P}_{ξ}^0 for $\mu_{\mathcal{P}}^{-1}(\xi)$ and call it a Spaltenstein variety.) Clearly $A_G(\xi)$, and hence $A_K(\xi)$, act on the set of irreducible components $\mathrm{Irr}(\mu_{\mathcal{P}}^{-1}(\xi))$. Fix $C \in \mathrm{Irr}(\mu_{\mathcal{P}}^{-1}(\xi))$, and consider $Z(C) := \overline{K \cdot C} \subset T^* \mathcal{P}$. Since $\xi \in \mathcal{N}_{\mathcal{P}}^{\theta} \subset \mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$, it follows from (2.5) that $Z(C)$ is in fact contained in the conormal variety

$$Z(C) \subset T_K^* \mathcal{P},$$

which is of course pure-dimensional of dimension $\dim(\mathcal{P})$. Hence

$$\dim(Z(C)) \leq \dim(\mathcal{P}).$$

But clearly

$$\dim(Z(C)) = \dim(K \cdot \xi) + \dim(C),$$

and thus

$$(2.11) \quad \dim(C) \leq \dim(\mathcal{P}) - \dim(K \cdot \xi).$$

Write $\mathrm{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$ for those irreducible components whose dimensions actually achieve the upper bound. (This set could be empty, for instance, as we shall see in Example 3.3 below when $\mathcal{P} = \mathcal{P}_{\beta}$ and ξ is a representative of a minimal nilpotent orbit. Note, however, that it is a general theorem of Spaltenstein's that if $\mathcal{P} = \mathcal{B}$, the full flag variety, then $\mathrm{Irr}_{\max}(\mu_{\mathcal{B}}^{-1}(\xi)) = \mathrm{Irr}(\mu_{\mathcal{B}}^{-1}(\xi))$.)

Proposition 2.12. Fix $\xi \in \mathcal{N}_{\mathcal{P}}^{\theta}$, set $\mathcal{O}_K = K \cdot \xi$, assume $\Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$ is nonempty, and fix $Q \in \Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$. Then

$$C(Q) := \overline{T_Q^* \mathcal{P}} \cap \mu_{\mathcal{P}}^{-1}(\xi)$$

is the union of elements in an $A_K(\xi)$ orbit on $\text{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$. The assignment $Q \mapsto C(Q)$ gives a bijection

$$(2.13) \quad \Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K) \longrightarrow A_K(\xi) \backslash \text{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi)).$$

Proof. Fix $C \in \text{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$. Then $\dim(Z(C)) = \dim(\mathcal{P})$ by definition. Notice that $Z(C)$ is nearly irreducible (and it is if K is connected). In general, the component group of K (which is finite by hypothesis) acts transitively on the irreducible components of $Z(C)$. But from the definition of $T_K^* \mathcal{P}$, the closure of each conormal bundle $T_Q^* \mathcal{P}$ consists of a subset of irreducible components of $T_K^* \mathcal{P}$ on which the component group of K acts transitively. Since $\dim(Z(C)) = \dim(T_K^* \mathcal{P})$, it follows that there is some Q such that

$$Z(C) = \overline{T_Q^* \mathcal{P}};$$

moreover Q must be an element of $\Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$. Clearly $Z(C) = Z(C')$ if and only if C and C' are in the same $A_K(\xi)$ orbit. The assignment $C \mapsto Q$ gives a bijection $A_K(\xi) \backslash \text{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi)) \rightarrow \Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$ which, by construction, is the inverse of the map in (2.13). This completes the proof. \square

Corollary 2.14. Let ξ_1, \dots, ξ_k be representatives of the K orbits on $\mathcal{N}_{\mathcal{P}}^{\theta}$. Then the map

$$Q \longrightarrow \left(\Phi_{\mathcal{P}}(Q), \overline{T_Q^* \mathcal{P}} \cap \mu_{\mathcal{P}}^{-1}(\xi_i) \right)$$

for i the unique index such that $K \cdot \xi_i$ dense in $\Phi_{\mathcal{P}}(Q)$ implements a bijection

$$K \backslash \mathcal{P} \longrightarrow \coprod_i A_K(\xi_i) \backslash \text{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi_i)).$$

Thus everything reduces to understanding the irreducible components of $\mu_{\mathcal{P}}^{-1}(\xi)$ of maximal possible dimension. For this we need some nontrivial results of Borho-MacPherson. [BM, Theorem 3.3] shows that the fundamental classes of the elements of $\text{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$ index a basis of $\text{Hom}_{W(\mathcal{P})}(\text{sgn}, \text{Sp}(\xi))$. Actually, to be precise, their condition for C to belong to $\text{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$ is that

$$\dim(C) = \dim(\mathcal{P}) - \frac{1}{2} \dim(G \cdot \xi).$$

To square with (2.11), we need to invoke the result of Kostant-Rallis [KR] that $K \cdot \xi$ is Lagrangian in $G \cdot \xi$. In any case, because $A_G(\xi)$ acts on $\text{Sp}(\xi)$ and commutes with the W action, $A_G(\xi)$ also acts on $\text{Hom}_{W(\mathcal{P})}(\text{sgn}, \text{Sp}(\xi))$, and [BM, Theorem 3.3] shows that this action is compatible with the action of $A_G(\xi)$ on $\text{Irr}(\mu_{\mathcal{P}}^{-1}(\xi))$. In particular this implies the following result.

Proposition 2.15. Fix $\xi \in \mathcal{N}_{\mathcal{P}}^{\theta}$. Then the number of $A_K(\xi)$ orbits on $\text{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$ equals the dimension of

$$\text{Hom}_{W(\mathcal{P})}(\text{sgn}, \text{Sp}(\xi)^{A_K}).$$

Combining Proposition 2.12 and 2.15, we obtain an alternate proof of Proposition 2.10 which makes no assumption on the connectedness of K .

Remark 2.16. The $\mathcal{P} = \mathcal{B}$ case of Corollary 2.14 is due to Springer (unpublished). In this case, $W(\mathcal{B})$ is trivial, and thus $\Phi_{\mathcal{B}}^{-1}(\mathcal{O}_K)$ has order equal to the W -representation $\text{Sp}(\xi)^{A_K}$.

It is of interest to compute the bijection of Corollary 2.14 as explicitly as possible. For instance, if $G_{\mathbb{R}} = \text{GL}(n, \mathbb{C})$ and $\mathcal{P} = \mathcal{B}$ consists of pairs of flags, the left-hand side of the bijection in Corollary 2.14 consists of elements of the symmetric group S_n . On the right-hand side, all A -groups are trivial, and the irreducible components in question amount to pairs of irreducible components of the

usual Springer fiber. Such pairs are parametrized by same-shape pairs of standard Young tableaux. Steinberg [St] showed that the bijection of the corollary amounts to the classical Robinson-Schensted correspondence.

A few other classical cases have been worked out explicitly ([vL], [Mc1], [T1], [T3]). But general statements are lacking. For instance, given Q and Q' , there is no known effective algorithm to decide if $\Phi_{\mathcal{P}}(Q) = \Phi_{\mathcal{P}}(Q')$. The next section is devoted to special cases of the parametrization which lead to nice general statements. It might appear that these special cases are too restrictive to be of much use. But it turns out that they encode exactly the geometry needed for the Adams-Barbasch-Vogan definition of Arthur packets. This is explained in Section 4.

3. \mathcal{P} -REGULAR K ORBITS

The main results of this section are Proposition 3.7(b) and Remark 3.10 which together give an effective computation of a portion of the bijection of Proposition 2.12 under the assumption that $\mu_{\mathcal{P}}$ is birational.

Definition 3.1 (see [ABV, Definition 20.17]). A nilpotent orbit \mathcal{O}_K of K on $\mathcal{N}_{\mathcal{P}}^{\theta}$ is called \mathcal{P} -regular (or simply regular, if \mathcal{P} is clear from the context) if $G \cdot \mathcal{O}_K$ is dense in $\mu_{\mathcal{P}}(T^*\mathcal{P})$. Since \mathcal{O}_K is Lagrangian in $G \cdot \mathcal{O}_K$ [KR], this condition is equivalent to

$$\dim(\mathcal{O}_K) = \frac{1}{2} \dim \mu(T^*\mathcal{P}) = \dim(\mathfrak{g}/\mathfrak{p}),$$

for any $\mathfrak{p} \in \mathcal{P}$. In other words, \mathcal{P} -regular nilpotent K -orbits meet the complex Richardson orbit induced from \mathfrak{p} . An orbit Q of K on \mathcal{P} is called \mathcal{P} -regular (or simply regular) if $\Phi_{\mathcal{P}}(Q)$ is a \mathcal{P} -regular nilpotent orbit. Note that regular \mathcal{P} -orbits need not exist in general (for instance, if $G_{\mathbb{R}}$ is compact and \mathcal{P} is not trivial).

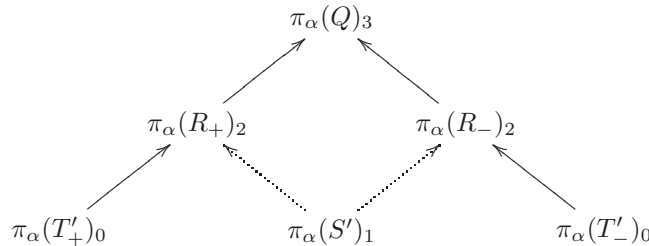
Since regular nilpotent K orbits are automatically maximal in the closure order on $\mathcal{N}_{\mathcal{P}}^{\theta}$, Proposition 2.6(2) shows that regular K orbits on \mathcal{P} are minimal in the weak closure order:

Proposition 3.2. *Suppose Q is a regular K orbit on \mathcal{P} . Then Q is minimal in the weak closure order on $K \backslash \mathcal{P}$.*

The next example shows that regular K orbits on \mathcal{P} need not be minimal in the full closure order (i.e. they need not be closed).

Example 3.3. Retain the notation of Example 2.7. Let \mathcal{P}_{α} (resp. \mathcal{P}_{β}) consist of parabolic subalgebras of type α (resp. β) and write π_{α} and π_{β} in place of $\pi_{\mathcal{P}_{\alpha}}$ and $\pi_{\mathcal{P}_{\beta}}$, and similarly for μ_{α} and μ_{β} . Then the closure order on $K \backslash \mathcal{P}_{\alpha}$ is obtained by the appropriate restriction from (2.8). (Subscripts now indicate dimensions; dashed edges are those covering relations present in the full closure order but not the weak one.)

(3.4)



The closure order on $K \backslash \mathcal{P}_\beta$ is again obtained by restriction from (2.8). (Once again subscripts indicate dimensions.)

$$(3.5) \quad \begin{array}{ccc} & \pi_\beta(Q)_3 & \\ & \uparrow & \\ & \pi_\beta(S)_2 & \\ \swarrow \cdots & & \nwarrow \cdots \\ \pi_\beta(T_+)_1 & & \pi_\beta(T_-)_1 \end{array}$$

In this case $\mathcal{N}_\alpha^\theta = \mathcal{N}_\beta^\theta$, and the closure order on $K \backslash \mathcal{N}_\mathcal{P}^\theta$ is just the bottom three rows of (2.9),

$$(3.6) \quad \begin{array}{ccccc} & 2_+^2 & & 2_+^1 2_-^1 & & 2_-^2 \\ & \swarrow & & \swarrow & & \swarrow \\ & 2_+^1 1_+^1 1_-^1 & & & & 2_-^1 1_+^1 1_-^1 \\ & \swarrow & & \swarrow & & \swarrow \\ & 1_+^2 1_-^2 & & & & \end{array}$$

From Proposition 5.2 below (for instance), both $\Phi_\alpha = \Phi_{\mathcal{P}_\alpha}$ and $\Phi_\beta = \Phi_{\mathcal{P}_\beta}$ are injective. There are enough edges in the weak closure order on $K \backslash \mathcal{P}_\alpha$ so that Proposition 2.6(1) allows one to conclude that Φ_α reverses the full closure order. In fact, Φ_α is the obvious order reversing bijection of (3.4) onto (3.6). Hence $\pi_\alpha(T'_\pm)$ and $\pi_\alpha(S')$ are \mathcal{P}_α -regular.

By contrast, Φ_β does not invert the dashed edges in (3.5): Φ_β maps $\pi_\beta(Q)$ to the zero orbit, and the three remaining orbits to the three orbits of maximal dimension in $\mathcal{N}_\mathcal{P}^\theta$. Hence $\pi_\beta(T'_\pm)$ and $\pi_\beta(S)$ are \mathcal{P}_β -regular. In particular, $\pi_\beta(S)$ is a \mathcal{P}_β -regular orbit which is not closed.

Finally note that the fiber of Φ_α over $2_\pm^1 1_+^1 1_-^1$ consists of a single element, while the corresponding fiber for Φ_β is empty. This is consistent with Proposition 2.10 since $\text{Sp}(\xi)$ (for ξ a representative of these orbits) is a one dimensional representation on which the simple reflection s_α (resp. s_β) acts nontrivially (resp. trivially). \square

An essential difference in the two cases considered in Example 3.3 is that μ_α is birational, but μ_β has degree two.

Proposition 3.7 ([ABV, Theorem 20.18]). *Suppose $\mu_\mathcal{P}$ is birational onto its image. Then:*

- (a) *Any regular K orbit on \mathcal{P} consists of θ -stable parabolic subalgebras (and hence is closed).*
- (b) *$\Phi_\mathcal{P}$ is a bijection from the set of regular K orbits on \mathcal{P} to the set of regular nilpotent K orbits on $\mathcal{N}_\mathcal{P}^\theta$.*

Proof. Fix a \mathcal{P} -regular nilpotent K orbit \mathcal{O}_K in $\mathcal{N}_\mathcal{P}^\theta$, $\xi \in \mathcal{O}_K$, and $Q \in \Phi_\mathcal{P}^{-1}(\mathcal{O}_K)$. Since $\mu_\mathcal{P}$ is birational, the set $\text{Irr}_{\max}(\mu_\mathcal{P}^{-1}(\xi))$ is a single point, and so Proposition 2.12 shows that Q is the unique orbit in $\Phi_\mathcal{P}^{-1}(\mathcal{O}_K)$. This gives (b).

Again since $\mu_\mathcal{P}$ is birational, there is a unique parabolic $\mathfrak{p} \in Q$ such that $\xi \in (\mathfrak{g}/\mathfrak{p})^*$. Since $\theta(\xi) = -\xi$, $\theta(\mathfrak{p})$ is also such a parabolic. So $\theta(\mathfrak{p}) = \mathfrak{p}$. Thus $Q = K \cdot \mathfrak{p}$ consists of θ -stable parabolic subalgebras. This gives the first part of (a). The same (well-known) proof of the fact that K orbits of θ -stable Borel subalgebras are closed (for example, [Mi, Lemma 5.8]), also applies to show that orbits of θ -stable parabolics are closed. (It is no longer true that a closed K orbit on \mathcal{P} consists of θ -stable parabolic subalgebras. But if a θ -stable parabolic algebra in \mathcal{P} exists, all closed orbits do indeed consist of θ -stable parabolic subalgebras.) \square

Because of the good properties in Proposition 3.7, we will mostly be interested in \mathcal{P} -regular orbits when $\mu_{\mathcal{P}}$ is birational. For orientation (and later use in Section 4) it is worth recalling a sufficient condition for birationality from [He]; see also [CM, Theorem 7.1.6] and [ABV, Lemma 27.8].

Proposition 3.8. *Suppose \mathcal{O} is an even complex nilpotent orbit. Let \mathcal{P} denote the variety of parabolic subalgebras in \mathfrak{g} corresponding to the subset of the simple roots labeled 0 in the weighted Dynkin diagram for \mathcal{O} (e.g. [CM, Section 3.5]). Then \mathcal{O} is dense in $\mu_{\mathcal{P}}(T^*\mathcal{P})$ and $\mu_{\mathcal{P}}$ is birational.*

□

Return to Proposition 3.7(a). Example 5.12 below shows that if $\mu_{\mathcal{P}}$ is birational, then not every (necessarily closed) K orbit of θ -stable parabolic subalgebras on \mathcal{P} need be regular. (A good example to keep in mind is the case when K and G have the same rank and $\mathcal{P} = \mathcal{B}$. Then the closed K orbits on \mathcal{B} parametrize discrete series representations with a fixed infinitesimal character. But the regular orbits are the ones which parametrize large discrete series.) So the question becomes: can one give an effective procedure to select the regular K orbits on \mathcal{P} from among all orbits of θ -stable parabolics (when $\mu_{\mathcal{P}}$ is birational)? This is only a small part of computing the parametrization of Corollary 2.14, so it is perhaps surprising that the answer we give after Proposition 3.9 depends on the power of the Kazhdan-Lusztig-Vogan algorithm for $G_{\mathbb{R}}$, the real form of G with complexified Cartan involution θ .

We need a few definitions. Recall that the associated variety of a two-sided ideal I in $U(\mathfrak{g})$ is the subvariety of \mathfrak{g}^* cut out by the associated graded ideal $\text{gr}I$ (with respect to the standard filtration on $U(\mathfrak{g})$) in $\text{gr}U(\mathfrak{g}) = S(\mathfrak{g})$. (From [BB1], if I is primitive, then $\text{AV}(I)$ is the closure of a single nilpotent coadjoint orbit.) Finally if \mathfrak{p} is a θ -stable parabolic subalgebra of \mathfrak{g} , recall the irreducible (\mathfrak{g}, K) -module $A_{\mathfrak{p}}$ constructed in [VZ]. (It would be more customary to denote these modules $A_{\mathfrak{q}}$, but we have already used the letter Q for another purpose.)

Proposition 3.9. *Suppose $\mu_{\mathcal{P}}$ is birational. Fix a closed K orbit Q on \mathcal{P} consisting of θ -stable parabolic subalgebras. Fix $\mathfrak{p} \in Q$. Then Q is \mathcal{P} -regular in the sense of Definition 3.1 if and only if*

$$\text{AV}(\text{Ann}(A_{\mathfrak{p}})) = \mu(T^*\mathcal{P}),$$

the closure of the complex Richardson orbit induced from \mathfrak{p} .

Remark 3.10. We remark that the condition of the proposition is effectively computable from a knowledge of the Kazhdan-Lusztig-Vogan polynomials for $G_{\mathbb{R}}$. More precisely, the results of Section 2 allow us to enumerate the closed orbits of K on \mathcal{P} from the structure of K orbits on \mathcal{B} . In turn, the description of $K \backslash \mathcal{B}$ has been implemented in the command `kbg` in the software package `atlas` (available for download from www.liegroups.org). Moreover, it is not difficult to determine which closed orbits consist of θ -stable parabolic subalgebras; in fact, if one of closed orbit does, then they all do. (Alternatively, one may implement the algorithms of [BH, Section 3.3], at least if K is connected.) For a representative \mathfrak{p} of each such orbit, one then uses the command `wcells`, to enumerate the cell of Harish-Chandra modules containing the Vogan-Zuckerman module $A_{\mathfrak{p}}$. (The computation of cells relies on computing Kazhdan-Lusztig-Vogan polynomials.) Finally $\text{AV}(\text{Ann}(A_{\mathfrak{p}})) = \mu(T^*\mathcal{P})$ if and only if the cell containing $A_{\mathfrak{p}}$ affords the Weyl group representation $\text{Sp}(\xi)^{A_G}$ (with notation as in Section 2), where ξ is an element of the Richardson orbit induced from \mathfrak{p} . Again, this is an effectively computable condition and is easy to implement from the output of `atlas`. Hence if $\mu_{\mathcal{P}}$ is birational, there is an effective algorithm to enumerate the \mathcal{P} -regular orbits of K on \mathcal{P} .

Remark 3.11. Suppose \mathcal{O} is an even complex nilpotent orbit, so that Proposition 3.8 applies. Then Proposition 3.7(b) shows that the algorithm of Remark 3.10 also enumerates the K orbits in $\mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^*$. Using the Kostant-Sekiguchi correspondence, this amounts to the enumeration of the real forms of \mathcal{O} , i.e. $G_{\mathbb{R}}$ orbits on $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}^*$. By contrast, if \mathcal{O} is not even, the only known way to enumerate the real forms of \mathcal{O} involves case-by-case analysis.

Proposition 3.9 is known to experts, but we sketch a proof (of more refined results) below; see also [ABV, Chapter 20]. We begin with some representation-theoretic preliminaries. Let $\mathcal{D}_{\mathcal{P}}$ denote the sheaf of algebraic differential operators on \mathcal{P} , and let $D_{\mathcal{P}}$ denote its global section. Since the enveloping algebra $U(\mathfrak{g})$ acts on \mathcal{P} by differential operators, we obtain a map $U(\mathfrak{g}) \rightarrow D_{\mathcal{P}}$. Let $I_{\mathcal{P}}$ denote its kernel, and $R_{\mathcal{P}}$ its image. By choosing a base-point $\mathfrak{p}_o \in \mathcal{P}$, it is easy to see that $I_{\mathcal{P}}$ is the annihilator of the irreducible generalized Verma module induced from $\mathfrak{p}_o \in \mathcal{P}$ with trivial infinitesimal character. We will be interested in studying Harish-Chandra modules whose annihilators contain $I_{\mathcal{P}}$, i.e. $(R_{\mathcal{P}}, K)$ -modules. For orientation, note that if $\mathcal{P} = \mathcal{B}$, $I_{\mathcal{B}}$ is a minimal primitive ideal, and thus any Harish-Chandra module with trivial infinitesimal character contains it.

Unlike the case of $\mathcal{P} = \mathcal{B}$, $U(\mathfrak{g})$ need not surject onto $D_{\mathcal{P}}$ in general, and so $R_{\mathcal{P}} \simeq U(\mathfrak{g})/I_{\mathcal{P}}$ is generally a proper subring of $D_{\mathcal{P}}$. Thus the localization functor

$$\begin{aligned} R_{\mathcal{P}}\text{-mod} &\longrightarrow \mathcal{D}_{\mathcal{P}}\text{-mod} \\ X &\longrightarrow \mathcal{X} := \mathcal{D}_{\mathcal{P}} \otimes_{R_{\mathcal{P}}} X. \end{aligned}$$

need not be an equivalence of categories. But nonetheless we have that the appropriate irreducible objects match. (Much more conceptual statements of which the following proposition is a consequence have recently been established by S. Kitchen.)

Proposition 3.12. *Suppose X is an irreducible $(D_{\mathcal{P}}, K)$ -module. Then its restriction to $R_{\mathcal{P}}$ is irreducible.*

Sketch. Irreducible $(D_{\mathcal{P}}, K)$ -modules are parametrized by irreducible K equivariant flat connections on \mathcal{P} . We show that the irreducible $(R_{\mathcal{P}}, K)$ -modules are also parametrized by the same set. The parametrizations have the property that support of the localization of either type of module parametrized by such a connection \mathcal{L} is simply the closure of the support of \mathcal{L} . This implies there are the same number of such irreducible modules and hence implies the proposition.

Let X be an irreducible $(R_{\mathcal{P}}, K)$ -module. Hence we may consider X as an irreducible (\mathfrak{g}, K) -module, say X' , whose annihilator contains $I_{\mathcal{P}}$. By localizing on \mathcal{B} , we may consider the corresponding irreducible K equivariant flat connection on \mathcal{B} , say \mathcal{L}' , parametrizing X' . The condition that $\text{Ann}(X') \supset I_{\mathcal{P}}$ can be translated into a geometric condition on \mathcal{L}' using [LV, Lemma 3.5], the conclusion of which is that \mathcal{L}' fibers over an irreducible flat K -equivariant connection on \mathcal{P} (with fiber equal to the trivial connection on \mathcal{B}_i). This implies that irreducible $(R_{\mathcal{P}}, K)$ -modules are also parametrized by K equivariant flat connections on \mathcal{P} , as claimed, and the proposition follows. \square

Remark 3.13. Proposition 3.12 need not hold when considering twisted sheaves of differential operators corresponding to singular infinitesimal characters.

Next suppose X is an irreducible $R_{\mathcal{P}}$ module. Let (\mathcal{X}^i) denote a good filtration on its localization \mathcal{X} compatible with the degree filtration on $\mathcal{D}_{\mathcal{P}}$. Let $\text{CV}(X)$ denote the support of $\text{gr}(\mathcal{X})$. This is well defined independent of the choice of filtration. Moreover, there is a subset $\text{cv}(X) \subset K \backslash \mathcal{P}$ such that

$$\text{CV}(X) = \bigcup_{Q \in \text{cv}(X)} \overline{T_Q^* \mathcal{P}}.$$

The set $\text{cv}(X)$ is difficult to understand, but there are two easy facts about it. First, if X is irreducible, there is a dense K orbit, say $\text{supp}_o(X)$ in the support of \mathcal{X} ; then $\text{supp}_o(X) \in \text{cv}(X)$. Moreover if $Q \in \text{cv}(X)$, then $Q \in \overline{\text{supp}_o(X)}$. So, for example, if $\text{supp}_o(X)$ is closed, then $\text{cv}(X) = \{\text{supp}_o(X)\}$.

Finally we define

$$\text{AV}(X) = \mu(\text{CV}(X)).$$

(Alternatively one may define $\text{AV}(X)$ as in [V3] without localizing. The fact that the two definitions agree follows from [BB3, Theorem 1.9(c)].) Clearly $\text{AV}(X)$ is the union of closures of K orbits on $\mathcal{N}_{\mathcal{P}}^{\theta}$. We let $\text{av}(X)$ denote the set of these orbits.

Here is how these invariants are tied together.

Theorem 3.14. *Retain the setting above. Then*

- (1) $\text{AV}(I_{\mathcal{P}}) = \mu(T^*\mathcal{P})$.
- (2) *If X is an irreducible $(R_{\mathcal{P}}, K)$ -module, then*

$$G \cdot \text{AV}(X) = \text{AV}(\text{Ann}(X)) \subset \text{AV}(I_{\mathcal{P}}).$$

Proof. Part (1) is Theorem 4.6 in [BB1]. The equality in part (2) is proved in [V3, Section 6]; the inclusion follows because X is an $R_{\mathcal{P}} = \text{U}(\mathfrak{g})/I_{\mathcal{P}}$ module. \square

Proposition 3.15. *Suppose X is an irreducible $(R_{\mathcal{P}}, K)$ -module such that there exists a \mathcal{P} -regular K orbit $Q \in \text{cv}(X)$. (For instance, suppose $\text{supp}_{\circ}(X)$ is \mathcal{P} -regular.) Then $\Phi_{\mathcal{P}}(Q)$ is a K orbit of maximal dimension in $\text{AV}(X)$; that is, $\Phi_{\mathcal{P}}(Q) \in \text{av}(X)$.*

Proof. Since $\text{AV}(X) = \mu(\text{CV}(X))$ and since $Q \in \text{cv}(X)$,

$$(3.16) \quad \Phi_{\mathcal{P}}(Q) \subset \text{AV}(X)$$

for any $(R_{\mathcal{P}}, K)$ -module. If Q is \mathcal{P} -regular, then the G saturation of the left-hand side of (3.16) is dense in $\mu(T^*\mathcal{P})$. But by Theorem 3.14 the right-hand side of (3.16) is also contained in $\mu(T^*\mathcal{P})$. So the current proposition follows. \square

Corollary 3.17. *Suppose X is an irreducible $(R_{\mathcal{P}}, K)$ -module. Then the following are equivalent.*

- (a) *there exist a \mathcal{P} -regular orbit $Q \in \text{cv}(X)$;*
- (b) *there exists a \mathcal{P} -regular orbit $\mathcal{O}_K \in \text{av}(X)$;*
- (c) $\text{Ann}(X) = I_{\mathcal{P}}$;
- (d) $\text{AV}(\text{Ann}(X)) = \text{AV}(I_{\mathcal{P}})$, i.e. $\text{AV}(\text{Ann}(X)) = \mu(T^*\mathcal{P})$.

Proof. The equivalence of (a) and (b) follows from the definitions above. Since the annihilator of any $R_{\mathcal{P}}$ module contains $I_{\mathcal{P}}$, the equivalence of (c) and (d) follows from [BKr, 3.6]. Theorem 3.14 and the definitions gives the equivalence of (b) and (d). \square

Proof of Proposition 3.9. If $\mathfrak{p} \in \mathcal{P}$ is a θ -stable parabolic, then the Vogan-Zuckerman module $A_{\mathfrak{p}}$ is the unique irreducible $(R_{\mathcal{P}}, K)$ -module whose localization is supported on the closed orbit $K \cdot \mathfrak{p}$ and thus, as remarked above, $\text{cv}(A_{\mathfrak{p}}) = \{K \cdot \mathfrak{p}\}$. So Proposition 3.9 is a special case of Corollary 3.17. \square

4. APPLICATIONS TO SPECIAL UNIPOTENT REPRESENTATIONS

The purpose of this section is to explain how the algorithm of Remark 3.10 produces special unipotent representations. Much of this section is implicit in [ABV, Chapter 27].

Fix a nilpotent adjoint orbit \mathcal{O}^{\vee} for \mathfrak{g}^{\vee} , the Langlands dual of \mathfrak{g} . Fix a Jacobson-Morozov triple $\{e^{\vee}, h^{\vee}, f^{\vee}\}$ for \mathcal{O}^{\vee} , and set

$$\chi(\mathcal{O}^{\vee}) = (1/2)h^{\vee}.$$

Then $\chi(\mathcal{O}^{\vee})$ is an element of some Cartan subalgebra \mathfrak{h}^{\vee} of \mathfrak{g}^{\vee} . There is a Cartan subalgebra \mathfrak{h} of \mathfrak{g} such that \mathfrak{h}^{\vee} canonically identifies with \mathfrak{h}^* . Hence we may view

$$\chi(\mathcal{O}^{\vee}) \in \mathfrak{h}^*.$$

There were many choices made in the definition of $\chi(\mathcal{O}^{\vee})$. But nonetheless the infinitesimal character corresponding to $\chi(\mathcal{O}^{\vee})$ is well-defined; i.e. $\chi(\mathcal{O}^{\vee})$ is well-defined up to G^{\vee} conjugacy and thus (via Harish-Chandra's theorem) specifies a well-defined maximal ideal $Z(\mathcal{O}^{\vee})$ in the center of $\text{U}(\mathfrak{g})$. We call $\chi(\mathcal{O}^{\vee})$ the unipotent infinitesimal character attached to \mathcal{O}^{\vee} .

By a result of Dixmier [Di], there exists a unique maximal primitive ideal in $\text{U}(\mathfrak{g})$ containing $Z(\mathcal{O}^{\vee})$. Denote it by $I(\mathcal{O}^{\vee})$, and let $d(\mathcal{O}^{\vee})$ denote the dense nilpotent coadjoint orbit in $\text{AV}(I(\mathcal{O}^{\vee}))$. The orbit $d(\mathcal{O}^{\vee})$ is called the Spaltenstein dual of \mathcal{O}^{\vee} (after Spaltenstein who first defined it in a different way); see [BV, Appendix A].

Fix $G_{\mathbb{R}}$ as above, and define

$$\text{Unip}(\mathcal{O}^{\vee}) = \{X \text{ an irreducible } (\mathfrak{g}, K) \text{ module} \mid \text{Ann}(X) = I(\mathcal{O}^{\vee})\}.$$

This is the set of special unipotent representations for $G_{\mathbb{R}}$ attached to \mathcal{O}^{\vee} . Since the annihilator of such a representation X is the maximal primitive ideal containing $Z(\mathcal{O}^{\vee})$, X is as small as the (generally singular) infinitesimal character $\chi(\mathcal{O}^{\vee})$ allows. These algebraic conditions are conjectured to have implications about unitarity.

Conjecture 4.1 (Arthur, Barbasch-Vogan [BV]). *The set $\text{Unip}(\mathcal{O}^{\vee})$ consists of unitary representations.*

We are going to produce certain special unipotent representations from the regular orbits of Definition 3.1. In order to do so, we need to shift our perspective and work on side of the Langlands dual \mathfrak{g}^{\vee} . So let $G'_{\mathbb{R}}$ be a real form of a connected reductive algebraic group with Lie algebra \mathfrak{g}^{\vee} and let K' denote the complexification of a maximal compact subgroup in $G'_{\mathbb{R}}$. Fix an *even* nilpotent coadjoint orbit \mathcal{O}^{\vee} . (This is equivalent to requiring that $\chi(\mathcal{O}^{\vee})$ is integral.) Define \mathcal{P}^{\vee} as in Proposition 3.8. Thus the main results of Section 3 are available in this setting.

Let X' denote an irreducible $(R_{\mathcal{P}^{\vee}}, K')$ -module, and let X denote the Vogan dual of X' in the sense of [V2]. Thus X is an irreducible Harish-Chandra module for a group $G_{\mathbb{R}}$ arising as the real points of a connected reductive algebraic group with Lie algebra \mathfrak{g} . Moreover, X has trivial infinitesimal character.

Recall that we are interested in representations with infinitesimal character $\chi(\mathcal{O}^{\vee})$. In order to pass to this infinitesimal character, we need to introduce certain translation functors. There are technical complications which arise in this setting for two reasons. First, $G_{\mathbb{R}}$ need not be connected (although it is in Harish-Chandra's class by our hypothesis). Second, $G_{\mathbb{R}}$ may not have enough finite-dimensional representations to define all of the translations one would like. Both of these complications disappear if we assume G is simply connected, and we shall do so here in the interest of streamlining the exposition. (It is of course possible to relax this assumption, as in [ABV, Chapter 27].)

Fix a representative $\rho \in \mathfrak{h}^*$ representing the trivial infinitesimal character. Choose a representative $\chi \in \mathfrak{h}^*$ representing the (integral) infinitesimal character $\chi(\mathcal{O}^{\vee})$ so that χ and ρ lie in the same closed Weyl chamber. Let $\nu = \rho - \chi$. Let F^{ν} denote the finite-dimensional representation of $G_{\mathbb{R}}$ with extremal weight ν ; this exists since we have assumed G is simple connected. Using it, define the translation functor $\psi = \psi_{\rho}^{\chi}$ (as in [KnV, Section VII.13]) from the category of Harish-Chandra modules with trivial infinitesimal character to the category of Harish-Chandra modules with infinitesimal character $\chi(\mathcal{O}^{\vee})$.

Theorem 4.2 (cf. [ABV, Chapter 27]). *Retain the notation introduced after Conjecture 4.1. In particular, fix an even nilpotent orbit \mathcal{O}^{\vee} , and let \mathcal{P}^{\vee} denote the variety of parabolic subalgebras corresponding to the nodes labeled 0 in the weighted Dynkin diagram for \mathcal{O}^{\vee} . Let X' be an irreducible $(R_{\mathcal{P}^{\vee}}, K')$ -module, assume G is simply connected, and let $Z = \psi(X)$ denote the translation functor to infinitesimal character $\chi(\mathcal{O}^{\vee})$ applied to the Vogan dual X of X' . Then the following are equivalent:*

- (a) Z is a (nonzero) special unipotent representation attached to \mathcal{O}^{\vee} .
- (b) there exists a \mathcal{P}^{\vee} -regular orbit $Q^{\vee} \in \text{cv}(X')$.

Proof. From the properties of the duality explained in [V2, Section 14] (and the translation principle), Z is nonzero with infinitesimal character $\chi(\mathcal{O}^{\vee})$ if and only if X' is annihilated by $I_{\mathcal{P}^{\vee}}$, i.e. if and only if X' descends to a $(R_{\mathcal{P}^{\vee}}, K)$ -module. Moreover Z is annihilated by a maximal primitive ideal if and only if the $R_{\mathcal{P}^{\vee}}$ -module X' has minimal possible annihilator, namely $I_{\mathcal{P}^{\vee}}$. The conclusion is that Z is special unipotent attached to \mathcal{O}^{\vee} if and only if X' is a $(R_{\mathcal{P}^{\vee}}, K)$ -module annihilated by $I_{\mathcal{P}^{\vee}}$. So the theorem follows from the equivalence of (a) and (c) in Corollary 3.17. \square

Since the duality of [V2] is effectively computable, and since the same is true of the translation functors ψ , the theorem shows Remark 3.10 translates into an effective construction of special unipotent representations. More precisely, one uses Remark 3.10 to enumerate the relevant \mathcal{P}^\vee -regular orbits, and for each one constructs the representation $X' = A_{\mathfrak{p}}$ of Proposition 3.9. As remarked in the proof of Proposition 3.9, X' satisfies condition (b) of Theorem 4.2. Applying the construction of the theorem gives special unipotent representations.

In fact, this construction may be understood further in light of the following refinement. In the setting of Theorem 4.2, fix a \mathcal{P}^\vee -regular orbit Q^\vee , and define $\mathbb{A}(Q^\vee)$ be the set of special unipotent representations attached to \mathcal{O}^\vee produced by applying Theorem 4.2 to all modules X' with $Q^\vee \in \text{cv}(X')$. Then the theorem implies

$$\text{Unip}(\mathcal{O}^\vee) = \bigcup \mathbb{A}(Q^\vee),$$

where the (not necessarily disjoint) union is over all \mathcal{P}^\vee -regular orbits.

The sets $\mathbb{A}(Q^\vee)$ are the Arthur packets defined in [ABV, Chapter 27]. While there are effective algorithms to enumerate $\text{Unip}(\mathcal{O}^\vee)$, there are no such algorithms for individual packets $\mathbb{A}(Q^\vee)$ (except in favorable cases). In any event, the discussion of the previous paragraph shows that *Remark 3.10 leads to an effective algorithm to enumerate one element of each Arthur packet of integral special unipotent representations*. These representatives are necessarily distinct.

5. EXAMPLES

Example 5.1 (Maximal parabolic subalgebras for classical groups). Suppose G is classical and \mathcal{P} consists of maximal parabolic subalgebra. Then it is well-known that

$$\text{ind}_{W(\mathcal{P})}^W(\text{sgn})$$

decomposes multiplicity freely as a W -module. Thus if $\text{Sp}(\xi)^{A_K}$ is irreducible as a W -module, then Proposition 2.10 implies $\Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$ is a single orbit. In particular if the orbits of $A_K(\xi)$ and $A_G(\xi)$ on irreducible components of the Springer fiber $\mu_{\mathcal{B}}^{-1}(\xi)$ coincide (for instance, if $A_K(\xi)$ surjects onto $A_G(\xi)$ for each ξ), then $\text{Sp}(\xi)^{A_K} = \text{Sp}(\xi)^{A_G}$ is irreducible and $\Phi_{\mathcal{P}}$ is injective.

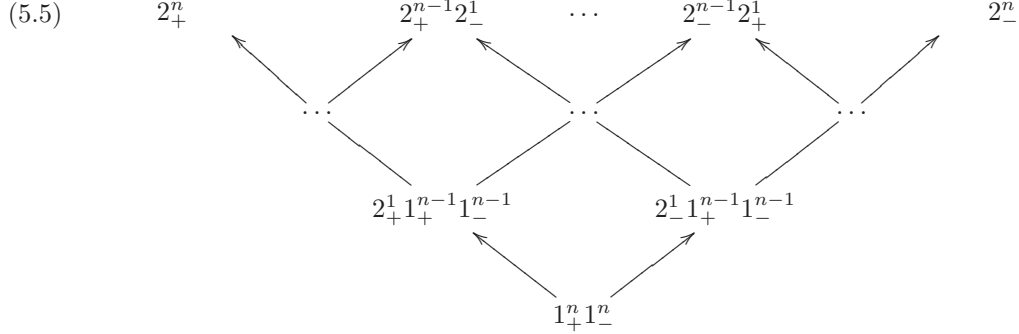
Proposition 5.2. *Suppose the real form $G_{\mathbb{R}}$ of G corresponding to θ is a classical semisimple Lie group with no complex factors whose Lie algebra has no simple factor isomorphic to $\mathfrak{so}^*(2n)$ or $\mathfrak{sp}(p, q)$. If \mathcal{P} consists of maximal parabolic subalgebras, then $\Phi_{\mathcal{P}}$ is injective.*

Proof. Unfortunately this follows from a case-by-case analysis of the classical groups. First note that the orbits of $A_K(\xi)$ and $A_G(\xi)$ on $\mu_{\mathcal{B}}^{-1}(\xi)$ are insensitive to the isogeny class of $G_{\mathbb{R}}$. So, by the remarks preceding the proposition, it is enough to examine when the two kinds of orbits coincide for a simply connected group $G_{\mathbb{R}}$ with simple Lie algebra. In type A, all A -groups are trivial (up to isogeny) so there is nothing to check. It follows from direct computation that $A_K(\xi)$ surjects on $A_G(\xi)$ for $G_{\mathbb{R}} = \text{Sp}(2n, \mathbb{R})$ and $\text{SO}(p, q)$, but that the image of $A_K(\xi)$ in $A_G(\xi)$ is always trivial for $\text{Sp}(p, q)$ and $\text{SO}^*(2n)$. This completes the case-by-case analysis and hence the proof.

Remark 5.3. For the groups in Proposition 5.2, the map $\Phi_{\mathcal{B}}$ is computed explicitly in [T1] and [T3]. Using Proposition 2.6(1) this gives one (rather roundabout) way to compute $\Phi_{\mathcal{P}}$ in these cases. For exceptional groups, the injectivity of the proposition fails. See Example 5.12 below.

Example 5.4. Suppose now $G_{\mathbb{R}} = \text{Sp}(2n, \mathbb{R})$ and \mathcal{P} consists of maximal parabolic of type corresponding to the subset of simple roots obtained by deleting the long one. (So if $n = 2$, $\mathcal{P} = \mathcal{P}_{\alpha}$

in Example 3.3.) Then the analysis of the preceding example extends to show that $\Phi_{\mathcal{P}}$ is an order-reversing bijection. The closure order on $K \backslash \mathcal{N}_{\mathcal{P}}^{\theta}$ (and hence $K \backslash \mathcal{P}$) is as follows.



Here, as before, we are using the parametrization of $K \backslash \mathcal{N}_{\mathcal{P}}^{\theta}$ given in [CM, Theorem 9.3.5]. There are thus $n + 1$ orbits which are \mathcal{P} -regular, all of which are closed according to Proposition 3.7(a) (which applies since \mathcal{P} is attached via Proposition 3.8 to the even complex orbit with partition 2^n).

In this setting, we may now apply Theorem 4.2. (Notationally the roles of the group and dual group must unfortunately be inverted: for the application, we should take $G^{\vee} = \mathrm{Sp}(2n, \mathbb{C})$ in the statement of the theorem.) Even though $\mathrm{SO}(n, n + 1)$ is not simply connected, the complications involving the relevant translation functors are absent, and the construction of the theorem nonetheless applies and produces $n + 1$ special unipotent representations for $\mathrm{SO}(n, n + 1)$.

Example 5.6. Suppose $G_{\mathbb{R}} = \mathrm{U}(n, n)$ and \mathcal{P} corresponds to the subset of simple roots obtained by deleting the middle simple root in the Dynkin diagram of type A_{2n-1} . Then $\Phi_{\mathcal{P}}$ is an order reversing bijection, and the partially ordered sets in question again look like that (5.5) using the parametrization of $K \backslash \mathcal{N}_{\mathcal{P}}^{\theta}$ given in [CM, Theorem 9.3.3]. Again there are $n + 1$ orbits which are \mathcal{P} -regular. The construction of Theorem 4.2 produces $n + 1$ special unipotent representation for $\mathrm{GL}(2n, \mathbb{R})$, each of which turns out to be a constituent of maximal Gelfand-Kirillov dimension in the degenerate principal series for $\mathrm{GL}(2n, \mathbb{R})$ induced from a one-dimensional representation of a Levi factor isomorphic to a product of n copies of $\mathrm{GL}(2, \mathbb{R})$.

In terms of representation theory of $G_{\mathbb{R}} = \mathrm{U}(n, n)$, it is well-known that the enveloping algebra in this case does surject on the ring of global differential operators on \mathcal{P} (e.g. the discussion of [T2, Remark 3.3]) and localization is an equivalence of categories. Because all Cartan subgroups in $\mathrm{U}(n, n)$ are connected, the only irreducible flat K -equivariant connections on \mathcal{P} are the trivial ones supported on single K orbits. The map $Q \mapsto \Phi_{\mathcal{P}}(Q)$ coincides with the map which sends the unique irreducible $(R_{\mathcal{P}}, K)$ -module supported on the closure of Q to the dense orbit in its (irreducible) associated variety, and is a bijection between such irreducible modules and the K orbits on $\mathcal{N}_{\mathcal{P}}^{\theta}$. It would be interesting to see if this observation could be used to give a geometric explanation of the computation of composition series of certain degenerate principal series for $\mathrm{U}(n, n)$ first given in [Sa] and later reproved in [Le]. (See, for instance, Sahi's module diagrams reproduced in [Le, Figure 7], for example.)

Example 5.7. Suppose $G_{\mathbb{R}} = \mathrm{Sp}(1, 1)$, a real form of $G = \mathrm{Sp}(4, \mathbb{C})$. If \mathcal{O} is the subregular nilpotent orbit for \mathfrak{g} and $\xi \in \mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^*$, then $A_K(\xi)$ is trivial, but $A_G(\xi) \simeq \mathbb{Z}/2$. So the proof of Proposition 5.2 does not apply. Let α denote the short simple root and β the long one. The closure order on

$K \setminus \mathcal{B}$ is given by

$$(5.8) \quad \begin{array}{c} Q \\ \beta \uparrow \\ R \\ \alpha \swarrow \quad \nwarrow \alpha \\ S_+ \quad S_- \end{array}$$

The picture for $K \setminus \mathcal{P}_\alpha$ is

$$(5.9) \quad \begin{array}{c} \pi_\alpha(Q)_3 \\ \uparrow \\ \pi_\alpha(R)_2 \end{array}$$

and for $K \setminus \mathcal{P}_\beta$

$$(5.10) \quad \begin{array}{ccc} & \pi_\beta(Q)_3 & \\ \swarrow & & \nwarrow \\ \pi_\beta(S_+)_2 & & \pi_\beta(S_-)_2 \end{array}$$

Here $\mathcal{N}_\alpha^\theta = \mathcal{N}_\beta^\theta = \mathcal{N}_\mathcal{B}^\theta$, and the closure order of K orbits is simply

$$(5.11) \quad \begin{array}{c} 2_+^1 2_-^1 \\ \uparrow \\ 1_+^2 1_-^2 \end{array}$$

in the notation of [CM, Theorem 9.3.5]. Then Φ_α is an order reversing bijection, but Φ_β is two-to-one over $2_+^1 2_-^1$. The reason is that

$$\mathrm{Sp}(\xi) = \mathrm{std} \oplus \chi,$$

where std is the two-dimensional standard representation of W and χ is a character on which the simple reflection s_α acts trivially and on which s_β acts nontrivially. The orbit $\pi_\alpha(R)$ is \mathcal{P}_α -regular, and the orbits $\pi_\beta(S_\pm)$ are \mathcal{P}_β -regular.

Example 5.12. As an example of what can happen in the exceptional cases, let G be the (simply connected) connected complex group of type F_4 and θ correspond to the split real form $G_\mathbb{R}$ of G . (So K is a quotient of $\mathrm{Sp}(3, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ by $\mathbb{Z}/2$.) Then the corresponding real form $G_\mathbb{R}$ is split. Let \mathcal{P} denote the variety of maximal parabolic obtained by deleting the middle long root from the Dynkin diagram, and let \mathcal{O} denote the corresponding Richardson orbit. Then \mathcal{O} is 40 dimensional and is labeled $F_4(A_3)$ in the Bala-Carter classification. Moreover \mathcal{O} is the unique orbit which is fixed under Spaltenstein duality. (Here we are of course identifying \mathfrak{g} and \mathfrak{g}^\vee .) For $\xi \in \mathcal{O}$, $A_G(\xi) = S_4$, the symmetric group on four letters. The weighted Dynkin diagram of \mathcal{O} has the middle long root labeled 2 and all others nodes labeled 0. So \mathcal{P} corresponds to \mathcal{O} as in Proposition 3.8.

From results of Djoković (recalled in [CM, Section 9.6]) there are 19 orbits of K on $\mathcal{N}_\mathcal{P}^\theta$. They are labeled 0–18; the orbit corresponding to label i will be denoted \mathcal{O}_K^i , and ξ^i will denote an element of \mathcal{O}_K^i . Orbits \mathcal{O}_K^{16} , \mathcal{O}_K^{17} , and \mathcal{O}_K^{18} are the three K orbits on $\mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^*$. From the discussion leading to [Ki, Table 2], it follows that $A_K(\xi^i)$ surjects onto $A_G(\xi^i)$ for $i = 0, \dots, 15$. In each of these cases, $A_G(\xi)$ is either trivial or $\mathbb{Z}/2$. We also have $A_K(\xi^{16}) = A_G(\xi^{16}) = S_4$. But $A_K(\xi^{17}) = D_4$, the dihedral group with eight elements, and $A_K(\xi^{17}) \rightarrow A_G(\xi^{17})$ is the natural inclusion into S_4 . Finally, $A_K(\xi^{18}) = \mathbb{Z}/2 \times \mathbb{Z}/2$ which injects into $A_G(\xi^{18})$.

For $i = 17$ and 18 , it is not immediately obvious how to read off $\mathrm{Sp}(\xi^i)^{A_K(\xi^i)}$ from, say, the tables of [Ca]. But for $i = 0, \dots, 16$, the component group calculations of the previous paragraph imply

that $\mathrm{Sp}(\xi^i)^{A_K(\xi^i)} = \mathrm{Sp}(\xi^i)^{A_G(\xi^i)}$, and such representations are indeed tabulated in [Ca]. Applying Proposition 2.10, it is then not difficult to show that

$$\#\Phi^{-1}(\mathcal{O}_K^i) = 1 \text{ if } i \in \{0, 1, 2, 3\} \cup \{9, 10, \dots, 16\}$$

and

$$\#\Phi^{-1}(\mathcal{O}_K^i) = 2 \text{ if } i \in \{4, 5, 6, 7, 8\}.$$

In more detail, the G -saturation of \mathcal{O}_K^4 and \mathcal{O}_K^5 is the complex orbit $A_1 \times \widetilde{A_1}$ in the Bala-Carter labeling, while \mathcal{O}_K^6 , \mathcal{O}_K^7 , and \mathcal{O}_K^8 have G saturation labeled by A_2 . The corresponding irreducible Weyl group representations in these two cases both appear with multiplicity two in $\mathrm{ind}_{W(\mathcal{P})}^W(\mathrm{sgn})$. All other relevant multiplicities are one.

We thus conclude that there are 22 orbits of K on \mathcal{P} which map via $\Phi_{\mathcal{P}}$ to some \mathcal{O}_K^i for $i = 0, \dots, 15$. Meanwhile, using the software program **atlas**, one can compute the closure order of K on \mathcal{B} , and thus (as explained in Section 2), the closure order on $K \backslash \mathcal{P}$. Figure 4.1 gives the full closure order for $K \backslash \mathcal{P}$. Vertices are labeled according to their dimension. (The edges in Figure 4.1 do *not* distinguish between the weak and full closure order. Doing so would make the picture significantly more complicated and difficult to draw.) There are thus 24 orbits of K on \mathcal{P} . Since 22 have been shown to map to \mathcal{O}_K^i for $i = 0, \dots, 15$, one concludes that the fiber of $\Phi_{\mathcal{P}}$ over \mathcal{O}^i for $i = 16$ and 17 must consist of just one element in each case.

In particular there are three \mathcal{P} -regular K orbits on \mathcal{P} which are bijectively matched via Proposition 3.7(b) to \mathcal{O}_K^{16} , \mathcal{O}_K^{17} , and \mathcal{O}_K^{18} . But from the **atlas** computation of the closure order on $K \backslash \mathcal{P}$, there are *four* closed orbits of K on \mathcal{P} . (These are in fact exactly the four orbits which are minimal in the weak closure order.) See Figure 5.12. The **atlas** labels of the closed orbits are 3, 22, 31, and 47. Their respective dimensions are 0, 1, 2, and 3. Applying the algorithm of Remark 3.10, one deduces that the three \mathcal{P} -regular orbits are 3, 31, and 47. Theorem 4.2 thus produces three distinct special unipotent representations, one in each of the three Arthur packets for $\mathcal{O} = d(\mathcal{O})$.

REFERENCES

- [ABV] Adams, J., D. Barbasch, and D. A. Vogan, Jr., *The Langlands Classification and Irreducible Characters for Real Reductive Groups*, Progress in Math, Birkhäuser (Boston), **104**(1992).
- [BV] D. Barbasch, D. A. Vogan, Unipotent representations of complex semisimple groups, *Ann. of Math. (2)*, **121** (1985), no. 1, 41–110.
- [BB1] W. Borho, J.-L. Brylinski, Differential operators on homogeneous spaces I. Irreducibility of the associated variety for annihilators of induced modules, *Invent. Math.*, **69** (1982), no. 3, 437–476.
- [BB3] W. Borho, J.-L. Brylinski, Differential operators on homogeneous spaces III. Characteristic varieties of Harish-Chandra modules and of primitive ideals, *Invent. Math.*, **80** (1985), no. 1, 1–68.
- [BKr] W. Borho and H. Kraft, Über die Gelfand-Kirillov Dimension, *Math. Ann.* **220** (1976), 1–24.
- [BM] W. Borho and R. MacPherson, Partial resolutions of nilpotent varieties, in *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, Astérisque **101** (1983), 2374.
- [BH] M. Brion and A. Helminck, On Orbit Closures of Symmetric Subgroups in Flag Varieties, *Canad. J. Math.*, **52** (2000), no. 2, 265–292.
- [Ca] R. Carter, *Finite Groups of Lie Type*, Wiley Interscience (New York), 1985.
- [CM] D. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold (New York), 1993.
- [Di] J. Dixmier, Idéaux primitifs dans l’algèbre enveloppante d’une de Lie semisimple complexe, *C. R. Acad. Sci. Paris (A)*, **271** (1970), 134–136.
- [Dj] D. Djoković, The closure order of nilpotent orbits for real forms of F_4 and G_2 , *J. Lie Theory*, **10** (2000), 471–490.
- [DR] J. M. Douglass and G. Röhrle, The Steinberg variety and representations of reductive groups, preprint, [arXiv:0802.0764v1](https://arxiv.org/abs/0802.0764v1).
- [He] W. Hesselink, Polarizations in classical groups, *Math. Z.*, **160** (1978), 217–234.
- [KL] D. Kazhdan and G. Lusztig, A topological approach to Springer’s representations, *Adv. in Math.*, **38** (1980), 222–228.
- [Ki] D. R. King, The component groups of nilpotents in exceptional simple real Lie algebras, *Comm. Algebra*, **20** (1992), no. 1, 219–284.

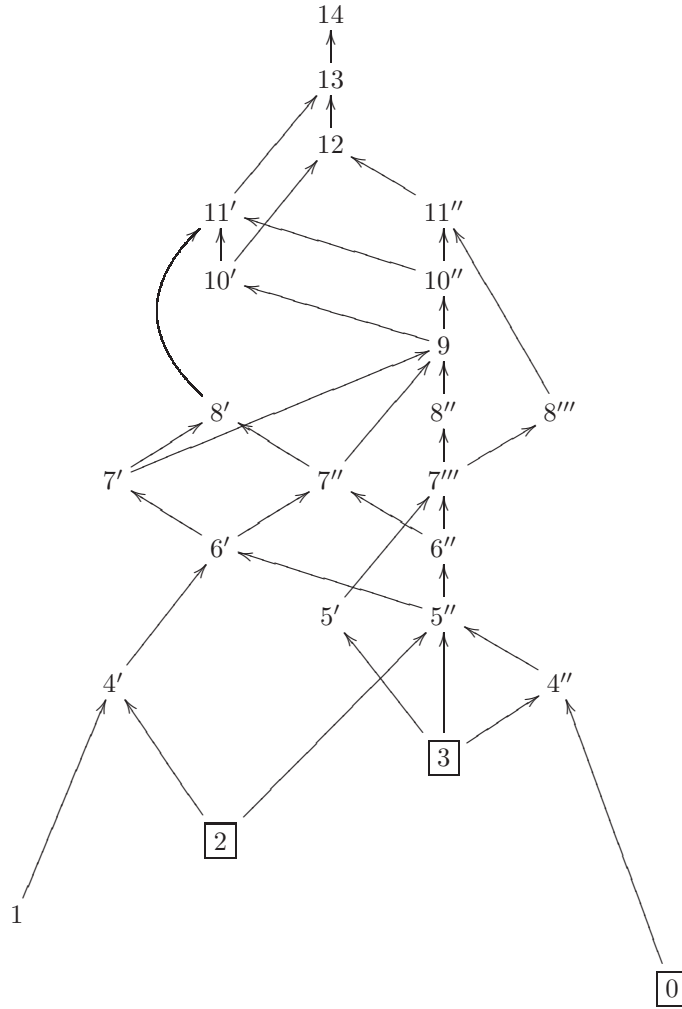


FIGURE 5.1. The full closure ordering of K -orbits on \mathcal{P} for $G_{\mathbb{R}} = F_4$ and $\mathcal{O} = F_4(A_3)$. Vertices are labeled according to their dimensions and boxed vertices are \mathcal{P} -regular. Note, in particular, that not every closed orbit is \mathcal{P} -regular.

- [KnV] A. W. Knap and D. A. Vogan, *Cohomological Induction and Unitary Representations*, Princeton Mathematical Series, number 45, Princeton University Press, 1995.
- [KR] B. Kostant and S. Rallis, Orbits and representations associated with symmetric spaces, *Amer. J. Math.*, **93** (1971), 753–809.
- [Le] S. T. Lee, On some degenerate principal series representations of $U(n, n)$. *J. Funct. Anal.*, **126** (1994), no. 2, 305–366.
- [LV] G. Lusztig and D. A. Vogan, Singularities of closures of K -orbits on flag manifolds, *Invent. Math.*, **71** (1983), no. 2, 365–379.
- [M] T. Matsuki, Closure relations for orbits on affine symmetric spaces under the action of minimal parabolic subgroups, *Representations of Lie groups, Kyoto, Hiroshima, 1986*, Adv. Stud. Pure Math., **14** (1988), 541–559.
- [Mc1] W. M. McGovern, On the Spaltenstein-Steinberg map for classical Lie algebras, *Comm. Algebra*, **27**(1999), no. 6, 2979–2993.
- [Mi] D. Milićić, Algebraic D-modules and representation theory of semisimple Lie groups, in *Analytic Cohomology and Penrose Transform*, Contemporary Mathematics, **154** (1993), 133–168.

- [R] W. Rossmann, Invariant eigendistributions on a semisimple Lie algebra and homology classes on the conormal variety II: Representations of Weyl groups, *J. Funct. Anal.*, **96** (1991), no. 1, 155–193.
- [RS] R. W. Richardson and T. A. Springer, The Bruhat order on symmetric varieties, *Geom. Dedicata*, **35** (1990), no. 1-3, 389–436.
- [Sa] S. Sahi, Unitary representations on the Shilov boundary of a symmetric tube domain, in *Representation theory of groups and algebras*, Contemp. Math., **145** (1993), 275–286.
- [St] R. Steinberg, An occurrence of the Robinson-Schensted correspondence, *J. Algebra*, **113** (1988), no. 2, 523–528.
- [T1] P. E. Trapa, Generalized Robinson-Schensted algorithms for real Lie groups, *Internat. Math. Res. Notices*, **1999**, no. 15, 803–834.
- [T2] P. E. Trapa, Annihilators and associated varieties of $A_q(\lambda)$ modules for $U(p, q)$, *Compositio Math.*, **129** (2001), no. 1, 1–45.
- [T3] P. E. Trapa, Symplectic and Orthogonal Robinson-Schensted algorithms, *J. Algebra*, **286** (2005), 386–404.
- [vL] M. van Leeuwen, A Robinson-Schensted algorithm in the geometry of flags for classical groups, Rijksuniversiteit Thesis, Utrecht, 1989.
- [V1] D. A. Vogan, Irreducible characters of semisimple Lie groups III. Proof of Kazhdan-Lusztig conjecture in the integral case, *Invent. Math.*, **71** (1983), no 2, 381–417.
- [V2] D. A. Vogan, Irreducible characters of semisimple Lie groups IV. Character-multiplicity duality, *Duke Math. J.*, **49** (1982), no. 4, 943–1073.
- [V3] D. A. Vogan, Associated varieties and unipotent representations, in *Harmonic analysis on reductive groups (Brunswick, ME, 1989)*, 315–388, Progress in Mathematics, **101**, Birkhäuser(Boston), 1991.
- [VZ] D. A. Vogan and G. Zuckerman, Unitary representations with nonzero cohomology, *Compositio Math.*, **53** (1984), no. 1, 51–90.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UT 84112, USA

E-mail address: ciubo@math.utah.edu

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY, SAKYO, KYOTO 606-8502, JAPAN

E-mail address: kyo@math.kyoto-u.ac.jp

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UT 84112, USA

E-mail address: ptrapa@math.utah.edu